

Generalized reliability-based syndrome decoding for LDPC codes

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Abstract

Aiming at bridging the gap between the maximum likelihood decoding (MLD) and the suboptimal iterative decodings for short or medium length LDPC codes, we present a generalized ordered statistic decoding (OSD) in the form of syndrome decoding, to cascade with the belief propagation (BP) or enhanced min-sum decoding. The OSD is invoked only when the decoding failures are obtained for the preceded iterative decoding method. With respect to the existing OSD which is based on the accumulated log-likelihood ratio (LLR) metric, we extend the accumulative metric to the situation where the BP decoding is in the probability domain. Moreover, after generalizing the accumulative metric to the context of the normalized or offset min-sum decoding, the OSD shows appealing tradeoff between performance and complexity. In the OSD implementation, when deciding the true error pattern among many candidates, an alternative proposed proves to be effective to reduce the number of real additions without performance loss. Simulation results demonstrate that the cascade connection of enhanced min-sum and OSD decodings outperforms the BP alone significantly, in terms of either performance or complexity.

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I. INTRODUCTION

For finite length low-density parity-check (LDPC) codes, the effective belief propagation (BP) decoding is regarded as suboptimal, owing to the presence of unavoidable short cycles in the bipartite graph representation of the code. On the other hand, due to the exponentially increased complexity with the block length, there exists no feasible maximum likelihood decoding (MLD) for practical LDPC codes.

To bridge the gap between BP and MLD decodings for short or medium length LDPC codes, a reliability-based order statistic decoding (OSD) was proposed to combine with the BP decoding [1] [2]. For such a BP-OSD reprocessing strategy, if no valid codeword is found at some iteration of the BP decoding, the delivered reliability information is used as the input to the OSD. Then the OSD decides whether one more iteration of the BP decoding is necessary according to some rule. For most codes of interest, the simulations have shown that BP decoding, in conjunction with order- p OSD reprocessing, yields near-optimal decoding performance when $p = 4$ [2]. Unfortunately, the incurred complexity in each OSD invoking boosts rapidly with the block length, hence greatly exceeds the complexity of one iteration of BP decoding [3] [4]. On the other hand, after approximating complex tanh function with simple min function for the check node updating, the min-sum algorithm [5] reduces the BP decoding complexity substantially. Furthermore, its enhanced variants, the normalized or offset min-sum algorithm [6], achieves comparable performance as the BP decoding.

Despite these efforts, there is still space to improve with respect to the tradeoff between performance and complexity. In [2], it was conjectured that, owing to the inaccurate reliability information delivered at the last iteration of BP decoding, order- p OSD in cascade connection with the BP would result in negligible improvement over the BP alone, where the word "cascade" implies one invoking of OSD per sequence in the paper. Nevertheless, it was shown in [7] that such cascade connection could achieve noticeable performance gain by drawing upon the accumulative log-likelihood ratio (LLR) information, instead of using only the LLR information

of the last iteration of BP decoding as mentioned in [2].

When the BP is implemented in probability domain, the available reliability information about each codeword bit is the probability of it being one or zero, we thus devise a similar accumulative probability metric, without converting it to the equivalent LLR representation, and hoping that it could achieve comparable performance as [7]. When the idea of accumulated reliability metric is applied in the context of reduced complexity decoding, say normalized or offset min-sum decoding, with respect to the BP alone, such a min-sum plus OSD decoding is justified by significant performance improvement and much less complexity.

The rest of the paper is organized as follows. Section II explains the steps of OSD based on various reliability metrics. Section III details the simulation results and discussion. Finally we conclude the work in Section IV .

II. IMPLEMENTATION OF THE SYNDROME DECODING

Assume a high rate binary (N, K) LDPC code with length N and dimension K . The parity check matrix is of the form $H_{M \times N}$, where $M = N - K$. The BPSK modulation maps one codeword $c = [c_1, c_2, \dots, c_N]$ into $x = [x_1, x_2, \dots, x_N]$ with $x_i = 2c_i - 1$, where $i = 1, 2, \dots, N$. After the symbols are transmitted through an additive white Gaussian noise (AWGN) memoryless channel, we obtain the corrupted sequence $y = [y_1, y_2, \dots, y_N]$ at the receiver, where $y_i = x_i + z_i$, z_i is an independent Gaussian random variable with a distribution of $\mathcal{N}(0, \sigma^2)$.

Given the bipartite graph of a code, the BP or enhanced min-sum decoding iteratively exchanges message between variable nodes and check nodes, the tentative hard decision is made after each iteration to test whether all check sums are satisfied. If so, exit immediately and declare decoding success. Otherwise continue the iterative decoding till the maximum number of iteration I_m is reached. The interested readers could refer to [6] [8] for more detailed description.

For the BP-OSD reprocessing structure mentioned in [2], to achieve near optimal performance, multiple invokings of OSD are required per sequence, which may result in unbearable decoding delay due to the high complexity attributed to the order- p OSD. While in the BP-OSD or min-sum-OSD cascade connection addressed in [7] and the paper, one invoking of OSD is needed only when no valid codeword is returned after the I_m th iterative decoding.

While in [2] [9] , the OSD is involved with the most reliable basis (MRB) of generator matrix G . In [10], it has been proved that the least reliable basis (LRB) of H and MRB of G are dual of

each other. Thus, the OSD, in the form of syndrome decoding, has equivalent error performance as that in [9]. After taking into account the lower dimension and sparseness of H , we prefer the OSD in the form of syndrome decoding, because it is easier to secure the LRB of H than the MRB of G .

When the BP is implemented in LLR domain, the accumulative reliability metric for variable node i is defined as [7]

$$r_i = \sum_{k=0}^{I_m} \alpha^{I_m-k} L_i^{(k)} \quad (1)$$

where α is a weight factor the optimal value of which may resort to the simulation. $L_i^{(k)}$ is the LLR output of variable node i at the k th iterative decoding, with

$$L_i^{(0)} = \ln \frac{P'(y_i|c_i = 1)}{P'(y_i|c_i = 0)} = \frac{2y_i}{\sigma^2}$$

where $P'(\cdot)$ is the conditional probability. Hence the hard-decision on variable node i is

$$\hat{c}_i = \begin{cases} 1 & \text{if } r_i > 0, \\ 0 & \text{if } r_i \leq 0. \end{cases}$$

Similarly, when the BP is in probability domain, assume the syndrome of the initial hard decision on y is \hat{c} , after each iteration of BP decoding, the accumulative probability metric for variable node i is defined as

$$q_i = \begin{cases} \sum_{k=0}^{I_m} \alpha^{I_m-k} P_i^{(k)} & \text{if } \hat{c}_i = 1, \\ I_m + 1 - \sum_{k=0}^{I_m} \alpha^{I_m-k} P_i^{(k)} & \text{if } \hat{c}_i = 0 \end{cases} \quad (2)$$

where $P_i^{(k)}$ is the probability of variable i being one at the k th iterative decoding.

Furthermore, when the normalized or offset min-sum decoding is substituted for the BP decoding, we extend the idea of accumulative reliability metric for the corresponding OSD as well.

$$u_i = \sum_{k=0}^{I_m} \alpha^{I_m-k} U_i^{(k)} \quad (3)$$

where $U_i^{(k)}$ is the reliability measurement of variable node i at the k th iterative decoding, with y_i being the initial $U_i^{(0)}$. The hard-decision on u_i for variable node i is defined as

$$\hat{c}_i = \begin{cases} 1 & \text{if } u_i > 0, \\ 0 & \text{if } u_i \leq 0. \end{cases}$$

Since the enhanced min-sum variants are known as uniformly most powerful (UMP) decoding, in the sense that no noise characteristic about channel is required, the same advantage is manifested for the OSD when drawing upon the definition of (3).

For the OSD, its information set includes all bits except those in the LRB. And it is possible for order- p OSD to solve the decoding failures when at most p erroneous bits are included in the information set. Hence, the construction of LRB and information set, dependent on the reliability evaluation of each codeword bit, determines the OSD performance. Given a specific code and a small number p , one reliability metric is said to be preferred to another in the sense that more decoding failures are within the scope of order- p OSD, considering that the estimated error pattern \hat{e} which satisfies the following discrepancy test [2] has a high probability to be the true error pattern.

$$D(y, \hat{e}) = \min \sum_{i: \hat{e}_i=1} |y_i| \quad (4)$$

Based on the decoding framework presented in [2], [7], [9], [10], with a modification only at step 4 below, the BP or enhanced min-sum variant, in cascade connection with the OSD, proceeds as follows.

- 1) The OSD is invoked iff all check sums are not satisfied after the I_m th iteration of BP or enhanced min-sum decoding.
- 2) With the reliability evaluation obtained with (1), (2) or (3), dependent on the iterative decoding method. Permutation λ_1 sorts the bits $\hat{e}_i, i \in [1, N]$ of error pattern \hat{e} in increasing order of the absolute reliability value, hence H is transformed into $H^{(1)}$ by column reordering. Permutation λ_2 on $H^{(1)}$ is to ensure the leftmost M columns of resultant $H^{(2)}$ to be independent, the indices of which form the LRB, and the other bit positions make up the information set. Accordingly, $\hat{e}^{(2)} = \lambda_2(\lambda_1(\hat{e}))$.
- 3) Apply Gaussian elimination on $H^{(2)}$ to transform it into systematic form. That is

$$EH^{(2)}\hat{e}^{(2)} = E\hat{c} \Rightarrow [I \ EH_{12}^{(2)}][\hat{e}_{11}^{(2)} \ \hat{e}_{12}^{(2)}]' = E\hat{c} \Rightarrow \hat{e}_{11}^{(2)} = EH_{12}^{(2)}\hat{e}_{12}^{(2)} + E\hat{c} \quad (5)$$

where E is the equivalent matrix for Gaussian elimination operation.

- 4) Evidently, $\hat{e} = \lambda_1^{-1}(\lambda_2^{-1}(\hat{e}^{(2)}))$. For order- p OSD, totally $\sum_{i=0}^{i=p} \binom{K}{i}$ candidate error patterns were supposed to be traversed. Conventionally, (4) will be solicited for each candidate to seek the one with the minimum discrepancy. To lower the complexity, we divide the task of seeking the optimal error pattern into two substeps.

- a) Assign a weight w_i for each bit i while transforming H into $H^{(1)}$ in step 2. Specifically, if bit i is in the j th position after reordering, it will be evaluated $w_i = j$. The use the metric $W_s = \sum_{i:\hat{e}_i=1} w_i$ to find β candidates which have the smallest W_s .
- b) Apply (4) to make a decision among the β error patterns. When a code has appropriate minimum distance, no performance loss is observed in the simulation even if $\beta = 1$.

In such a way, many real additions in computing (4) otherwise are replaced by simple integer additions.

- 5) Apply modulo 2 addition of \hat{e} with \hat{c} to recover the original codeword.

III. SIMULATION RESULTS AND DISCUSSION

The simulations are performed on a number of LDPC codes, but we will present only the results for (504, 252) [11]. At each SNR point, at least 100 decoding failures are detected. To manage computational complexity of the OSD, the simulations are limited to at most order-2 OSD. (1), (2) and (3) all relate to the evaluation of parameter α . As reported in [7], the optimal value of α relies on I_m and the code itself. For (504, 252), it was found $\alpha = 1$ results in the best decoding performance when $I_m = 20$. Therefore, For the sake of simplicity, $\alpha = 1$ is assumed in the simulations below.

The frame error rate (FER) curves for each iterative decoding, all with $I_m = 20$, in cascade connection separately with order-0, 1, 2 OSD, are depicted in Fig. 1, also included are FER curves for the BP decoding with $I_m = 20, 100$, respectively. In the legend, BP+J-0, 1, 2 denotes the BP cascaded separately with order-0, 1, 2 OSD, whose reliability metric is given by (1). While for the OSD in BP+L-0, 1, 2, its reliability metric is given by (2). (3) is used in N-Ms+L-0, 1, 2 and O-Ms+L-0, 1, 2, where "N-Ms" stands for the normalized min-sum, and "O-Ms" the offset min-sum.

As seen in Fig. 1 and Fig. 2, Whatever in the FER or bit error rate (BER) metric, given a specific p , there exists no performance difference among each BP/Min-sum plus order- p OSD combination. Specifically, in Fig. 1, the BP+J-0, BP+L-0, N-Ms+L-0 and O-Ms+L-0, indicated as Order-0 group, all achieve 0.5 dB at FER=10⁻³ over the BP ($I_m = 20$) and 0.12 dB over the BP ($I_m = 100$). While the Order-1, 2 groups outperform the BP ($I_m = 100$) by 0.35, 0.50 dB, respectively.

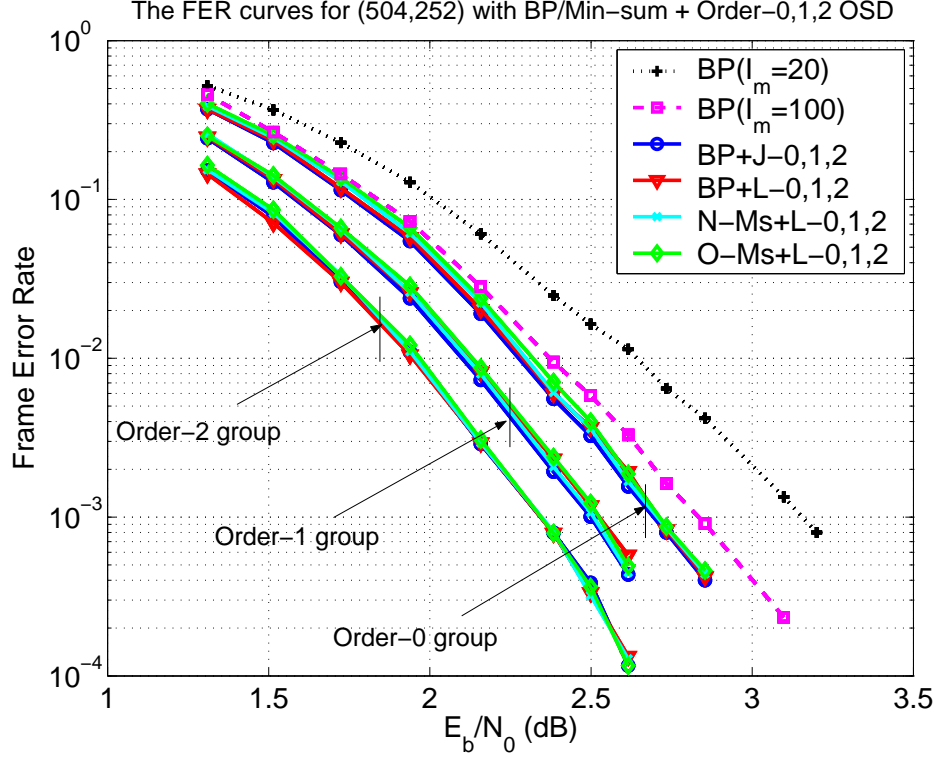


Fig. 1 The FER curves for (504,252) with BP/Min-sum + Order-0,1,2 OSD decoding schemes of various reliability metrics

From the perspective of BER, as shown in Fig. 2, at $\text{BER}=10^{-4}$, Order-2 group achieves 0.65, 0.45 dB over the BP with $I_m = 20, 100$, respectively. While Order-0, 1 groups achieve observable performance improvement over the BP alone as well, the BER curves of which are not depicted in Fig. 2.

The OSD always returns an valid codeword, nevertheless this codeword may contain more erroneous bits than the decoding failure when it is not the correct one. Therefore, the performance gain achieved in BER metric is not so striking as in FER metric assuming the same decoding combination, as reported in [4] as well.

Denote the average number of iterations as A_{ni} , it is shown in Fig. 3 that in the SNR region of interest, the A_{ni} required for each decoding combination is less than that of the BP ($I_m = 100$) alone. Taking into account Fig. 1, Fig. 2 and Fig. 3, we find that the normalized or offset min-sum, as a reduced complexity decoding, when in cascade connection with the order- p OSD, appears to be the most competitive decoding scheme, in terms of tradeoff between performance and complexity.

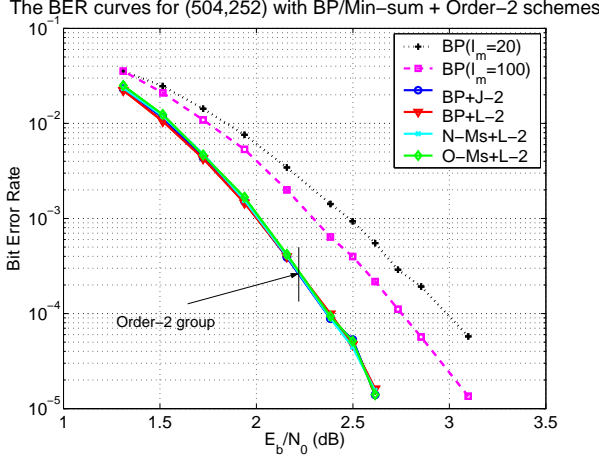


Fig. 2 The FER curves for (504,252) with BP/Min-sum + Order-2 OSD decoding schemes of various reliability metrics

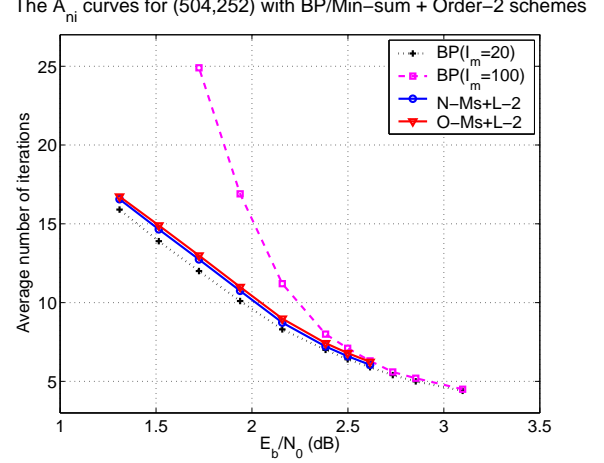


Fig. 3 The A_{ni} curves for (504,252) with BP/Min-sum + Order-2 OSD decoding schemes of various reliability metrics

For the detailed complexity analysis about the BP in probability and LLR domain, and normalized or offset min-sum, the interested readers could refer to [2] [6]. Among these four schemes, the offset min-sum requires the least complexity. To obtain the reliability evaluation for all codeword bits, NI_m real additions are required during iterative decoding to accumulate it. As far as the proposed OSD is concerned, the binary operations for Gaussian elimination is of the order $O(N^3)$, the integer and binary operations for each phase- l ($1 \leq l \leq p$) of order- p OSD is of the order $O(N^{l+1})$. Also included are $N \log_2 N$ real additions for sorting codeword bits and $\beta\gamma$ real additions for estimating the error pattern, where γ is the average number of nonzero element in an error pattern.

IV. CONCLUSIONS

In this paper, we have generalized the BP-OSD postprocessing framework into more applications. For the order- p OSD, we extend the accumulated reliability metric which was applied in the LLR domain of BP decoding, to the probability domain of BP decoding. Furthermore, an extension of accumulated metric to the reduced complexity decoding, say min-sum variants, has shown that such combination will be the most advantageous, in the sense that no channel characteristic is required and the best tradeoff is achieved between performance and complexity.

Given a fixed p , the performance of an order- p OSD will suffer from the block length increment. How to mitigate such degradation remains to be solved in the future work.

ACKNOWLEDGMENT

Sincere thanks to R.Neal for his online software [12]. This work was supported by National Science Foundation of China under Grant No.60472104.

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